ECONOMIC DESIGN OF MOVING AVERAGE CONTROL CHARTS TO MAINTAIN CURRENT CONTROL OF A PROCESS MEAN

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ECONOMIC DESIGN OF MOVING AVERAGE CONTROL CHARTS

TO MAINTAIN CURRENT CONTROL OF A PROCESS MEAN

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TO MY MOTHER AND FATHER

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SUMMARY

The purpose of this investigation was to develop an expected minimum cost quality control model for the moving average control chart to control the mean of a normally distributed process. This investigation assumes a process which has one in control state and one out of control state and that the time to shift out of control follows the exponential distribution. This was accomplished by treating the transition of the process between states as a finite Markov chain. A transition matrix was used to calculate the steady state probabilities of the process being in either state, and the expected cost of the quality control procedure was calculated from these. The solution presents the minimum cost sample size, interval between samples, critical region parameter and moving average index.

A numerical solution procedure is developed and programmed for a digital computer to determine the minimum cost combination of parameters. Numerical examples with various model parameters and cost coefficients are investigated using this procedure, and the optimal values of the test parameters and expected cost are tabulated. Sensitivity of the optimal test parameters to change in cost coefficients and system parameters is also investigated, and the cost of using a moving average chart compared with the cost of using an $\overline{X}$ chart.

The results of this investigation indicate that the cost of using a moving average control chart is sensitive to small changes in the cost coefficients of the system being modelled, and that the moving average chart is more economical to use than the $\overline{X}$ chart in every case considered.

CHAPTER I

PROBLEM DESCRIPTION

This chapter will present an overview of control charts and moving average procedures. A statement of the problem to be investigated and a review of the literature related to this problem will also be included.

1.1 Statistical Quality Control

One fundamental concept underlying all of statistical quality control is the idea that in any manufacturing process, the quality of the output is subject to chance variation which can never be completely eliminated. In addition to this chance variation, the quality of the output is subject to other causes of variation, termed assignable causes, as they can be traced back to a particular source. When all of the assignable causes of variation have been eliminated, the process is said to be in a state of statistical control; otherwise it is said to be out of control. A control chart is one of the tools of statistical quality control and is widely used for monitoring manufacturing processes.

The procedure that is followed when using a control chart is to take a random sample of a certain size at regular intervals and to measure the quality of the specific units sampled. A test statistic is then computed and plotted on the control chart. The value of this statistic can fall into one of two categories. First, it may be a value which is judged likely to occur if the process is in control. This corresponds to the area on the chart between the upper and lower control limits, and if the statistic falls within this region, the process is allowed to continue in operation. Alternatively, it may have a value which is judged unlikely to occur if the process is in control. The set of these "unlikely" values constitutes what is called the test critical region and corresponds to the area on the chart above the upper control limit and below the lower control limit. If the value of the test statistic falls in these areas, a search is made for the cause of the variation. If the determination of the quality of the units sampled is made on the basis of comparison with a standard, then the sampling procedures is said to be one for attributes; if the determination is made by measuring some aspect of the quality on a numerical scale, the sampling procedure is said to be one for measurements. Attributes sampling is usually performed in conjunction with a fraction defective control chart (p chart) or a defects per unit control chart (c chart). Measurements sampling is usually performed in conjunction with a control chart for the process mean (called an $\overline{X}$ chart) and a control chart for process variability (called an R chart or a σ chart, depending on how the process variability is estimated). In order to obtain equivalent protection for a specified shift, the p chart must generally use a larger sample size than an $\overline{X}$ chart. Further information on control charts is available in Duncan (3) and Grant and Leavenworth (8).

1.2 Survey of the Literature

The pioneering work in the field of statistical quality control was done by Shewart (14), who introduced the idea of a chart to visually record the progress of some quality characteristic. The approach used did not explicitly take into consideration the costs of both type I and type II errors. Rather than this, the method was to select a sample size that would detect a given shift in the process with a prescribed power. Frequently, sample sizes of four or five were used in conjunction with control limits set at ±3 standard deviations of the test statistic, and any convenient sampling intervals.

The first work which explicitly considered the costs of both type I and type II errors was reported by Duncan (4). He constructed an economic model of a control chart procedure where a single assignable error exists. By manipulating his model, he was able to find the optimal sample size, sampling interval and control limits. His model was of the form:

$$\text{Profit} = \text{Income-Cost}$$

Since he assumed that income is independent of cost, maximization of profit was equivalent to the minimization of cost. Duncan's model assumed that all shifts in the process are by a fixed amount, with the average time between shifts being $\frac{1}{\lambda}$ hours and where the probability of shifting out of control between time t and t+ Δt is $\lambda e^{-\lambda t}\,\Delta t$, for small Δt. Goel, Jain and Wu (7) developed an algorithm for finding the optimal system parameters in Duncan's Model. Their algorithm consists of solving an implicit equation in sample size, sampling interval and critical region parameter.

Cowden (2) developed a model for a test procedure for controlling the mean of a process. His model has the form:

$$C = C_1 + C_2 + C_3$$

where C_1 is the operating cost of the test procedure, C_2 is the cost of investigating the process and C_3 is the cost of producing defectives. His model employs a number of assumptions which would seem to limit its applicability. These assumptions are that the process starts every day in the out of control state, that once the trouble is detected, it is quickly corrected and no further trouble can occur that day, that the cost of looking for trouble is proportional to the shift in the mean, and that the probability of finding trouble is a function of the cost of the search.

Knappenberger and Grandage (10) developed a model to minimize the long term expected cost per unit of a quality control procedure. Their model has the form:

$$C = C_1 + C_2 + C_3$$

where C_1 is the cost of sampling and testing, C_2 is the cost of investigating the process when the value of the test statistic falls in the critical region, and C_3 is the cost of producing defectives. They assume that the process parameter is a continuous random variable that can be approximated by a discrete random variable and that the process parameter has one in control state and multiple out of control states. When the process goes out of control it is assumed to stay out of control until it is detected, and if the process shifts from one out of control state to another, it can only shift further out of control.

Duncan (5) extended his earlier work to include the case where several assignable errors are allowed to occur. He indicated that the increased accuracy of multiple assignable errors is often wasted by inaccurate estimation of the cost coefficients.

Montgomery and Klatt (12) have developed an economic model of a quality control procedure based on Hotelling's T^2 control chart, which is the multivariate analog of the $\overline{X}$ chart. Mance (11) has developed a model of a quality control procedure based on the fraction defective control chart. These authors used models which are similar to Knappenberger and Grandage's.

Baker (1) has done a study of two alternative process models for an $\overline{X}$ chart. His model assumes that the process begins operation in the in control state, and that shifts out of control occur at the beginning of the period. Sampling is done at the end of the period, as is any corrective action, so that when the process shifts out of control it will stay out of control for at least one period. Baker's results indicate that the assumption of exponential shifts from the in control state to the out of control state, as made by Knappenberger and Grandage and others, may lead to poor results. Heikes, Montgomery and Yeung (7) have adapted Baker's economic model to a multivariate quality control procedure where the shifts from the in control state to the out of control state follow the geometric, the Poisson and the logarithmic series distributions. Their results indicate that the assumption of Markovian shifts and the shape of the distribution of time to failure are important to the determination of optimal test parameters.

Taylor (16) studied the case of the $\overline{X}$ chart where one in control and one out of control state are allowed, where the variance is known and where normality is assumed. He found that sample size and sampling interval should be determined at each stage of the process by calculating current posterior probabilities, and that a fixed sample size/sampling interval approach will lead to non-optimal results. Despite his findings, most work in the field of economic design of control procedures continues to use fixed sample sizes and sample intervals, because of their greater simplicity.

The literature of the economic design of quality control procedures makes no mention of any control chart which bases its definition of control status on data taken from more than one period. Such charts, however, are discussed from a statistical point of view and their properties are compared with single period control charts.

The first mention of multi-period control charts is made by Roberts (13). He describes a geometric moving average control procedure to control the mean of a process, and he shows how to construct moving average control charts using a graphical procedure. In a later paper (14), Roberts compares the performance of the $\overline{X}$ chart, the moving average chart, the geometric moving average chart, the cumulative sum chart, and the chart for the Girshick-Rubin test (which plots a linear combination of the current $\overline{X}$ and the natural logarithm of the quantity: one plus the exponential of the test statistic from the previous period).

He assumes that the observations follow the normal distribution, and that the standard deviation is known. He uses a fixed sample size and sampling interval. He compares the charts on the basis of a graph of the expected number of periods which pass before a shift is detected versus the size of the shift. The Girshich and Rubin test is found to have the best performance.

Wortham and Heinrich (17) describe the application of exponential smoothing techniques to mean and variance control charts. They show that the exponential smoothing mean control chart shows more detail than the traditional Shewart $\overline{X}$ chart since a control point can be generated for each of the data points.

1.3 Statement of the Problem to be Investigated

A study of the literature mentioned in the previous section shows that multi-period control charts tend to be inherently more complicated than single period charts. For this reason, a decision was made to select a multi-period control chart that was as simple as possible and yet which would have some practical applications. The moving average control chart for the process mean, in conjunction with the Knappenberger and Grandage economic model (adapted to the case of a single out of control state) was chosen on this basis. The purpose of this investigation is to develop a procedure to minimize the long term expected cost per unit associated with the use of such a chart. This procedure will specify the optimal moving average index (M), sample size (N), sampling interval (K), and critical region parameter (α).

The costs of using such a chart are closely related to probabilities of making two types of errors. One kind, called type I error, is made when the value of the test statistic falls in the critical region (that is, above the upper control limit (UCL) and below the lower control limit (LCL)), and yet the process is still in control. When this error is made, the costs of unnecessary investigation and/or loss of production are incurred. The other kind, called type II error, is made when the value of the test statistic falls outside the critical region and yet the process is in fact out of control. This error incurs the cost of bad output. The probabilities of making both of these errors can be reduced by increasing the sample size and the sampling frequency. In addition, the probability of making a type I error can be reduced by decreasing the size of the critical region (moving the UCL and LCL further apart). A smaller critical region makes it less likely that a test statistic computed from a sample drawn from an in control process will fall above the UCL and below the LCL. On the other hand, a smaller critical region also makes it easier for a test statistic computed from an out of control process to fall between the UCL and LCL, so the probability of type II error is increased. The size of the moving average index (M) also influences the probability of type II error. With larger values of M, the sensitivity of the model to a shift from the in control state to the out of control state tends to decrease, since the effect of the shift is masked by those terms in the average which come from an in control process. The more terms there are, the greater the masking. Thus large values of M increase the probability of type II error. This effect is counterbalanced by the reduction in variance of the test statistic which occurs when larger values of M are used. The variance decreases as M increases because the moving average index appears in the denominator of the expression for the variance: $\sigma_T^2 = \frac{\sigma^2}{MN}$. A smaller variance implies a smaller probability of type II error, for a given probability of type I error. The overall effect of the size of M on the probability of type II error depends on the parameters of the process being controlled.

It is assumed that the procedure described will be applicable to a wide variety of processes. In each case, however, the actual value of the parameters M, N, K, and α will depend on the accurate estimation of various cost coefficients and other system parameters. This investigation will include a study of how variation in these quantities affects the behavior of the mathematical model.

In this investigation we will assume that the quality characteristic is normally distributed, with a known standard deviation, and that the process mean will be at one of two levels: 1) a level associated with the in control state, μ_0, and 2) a level which is associated with the out of control state, μ_1, and it is assumed that the difference between these levels is known in terms of the standard deviation of the process.

The chapters which follow include a description of the moving average control chart, the development of an economic model, solution techniques and numerical examples.

CHAPTER II

MATHEMATICAL MODEL

This chapter presents a description of the moving average control chart for the process mean and the development of a mathematical model of its operation.

2.1 The Moving Average Control Chart for the Process Mean

The moving average chart for the process mean is generally similar to the standard, single period, $\overline{X}$ chart. The procedure usually employed in conjunction with an $\overline{X}$ control chart consists of taking a random sample of N units after every K units have been produced, measuring the quality on a numerical scale, and computing the test statistic $\overline{X}_t$:

$$\overline{X}_t = \frac{1}{N} \sum_{i=1}^{N} X_{ti}$$

where the X_{ti} are the individual observations taken in period t. The test statistic $\overline{X}_t$ is then plotted on the control chart. The chart is characterized by a center line, which corresponds to the average quality of the output, and two other lines, one called the upper control limit (UCL) and the other called the lower control limit (LCL). If we assume that the quality characteristic which we are measuring follows the normal distribution, and that the mean μ and standard deviation σ are known, then the probability is $1 - \alpha$ that the mean will fall between $\mu + Z_{\alpha/2} \frac{\sigma}{\sqrt{N}}$

and $\mu - Z_{\alpha/2} \frac{\sigma}{\sqrt{N}}$. The probability α is called the critical region parameter or the probability of type I error.

When an $\overline{X}$ chart is used in conjunction with a moving average procedure, the test statistics $\overline{X}_t$ from a number of periods are used to form a moving average test statistic T_t. If M is the number of terms in the moving average, then T_t is defined as follows:

$$T_t = \frac{1}{M}(\overline{X}_{t-M+1} + \overline{X}_{t-M+2} + \ldots + \overline{X}_t)$$

The above definition assumes that the moving average control chart has been in operation for at least M periods since the last out of control T_t was plotted. When this condition is met, the system will be described as being in steady state operation.

The behavior of a moving average is dependent on M, the number of terms it includes. Basically, there are three types of changes in a system which can influence a moving average: a sudden, temporary change, (random shock), a sudden, permanent change (step increase or decrease), and a gradual trend, either upward or downward. Since it is assumed that the process will not correct itself, this investigation will only be concerned with sudden, permanent shifts in the process mean. Under the influence of such an input, the mean of the test statistic will change steadily for M periods, after which the mean will be equal to the actual value of the system.

The development of the mathematical model is described in the following sections.

2.2 General Form of the Model

The mathematical model of the moving average control procedure will be assumed to consist of the sum of three expected costs, and will be written as:

$$E(C) = E(C_1) + E(C_2) + E(C_3)$$

where $E(C_1)$ is the expected cost per unit associated with sampling and testing, $E(C_2)$ is the expected cost per unit of investigating and correcting the process when the test statistic falls in the critical region, and $E(C_3)$ is the expected cost per unit of producing defectives.

2.2.1 Expected Cost of Sampling and Testing

It will be assumed, as previous authors have done (4), (10), that the expected cost of sampling and testing has two components: a fixed cost, independent of the sample size, and a cost that varies with the number of units sampled. Both cost components are divided by the number of units produced between successive samples to obtain an average cost per unit of sampling and testing. Since all terms are assumed constant,

$$E(C_1) = \frac{a_1 + a_2N}{K}$$

Where a_1 = fixed cost per sample

a_2 = cost per unit sampled

N = number of units sampled

K = number of items produced between the start of the current sampling to just prior to the start of the subsequent In Figure 1 below, K is the number of units produced between points a and c.

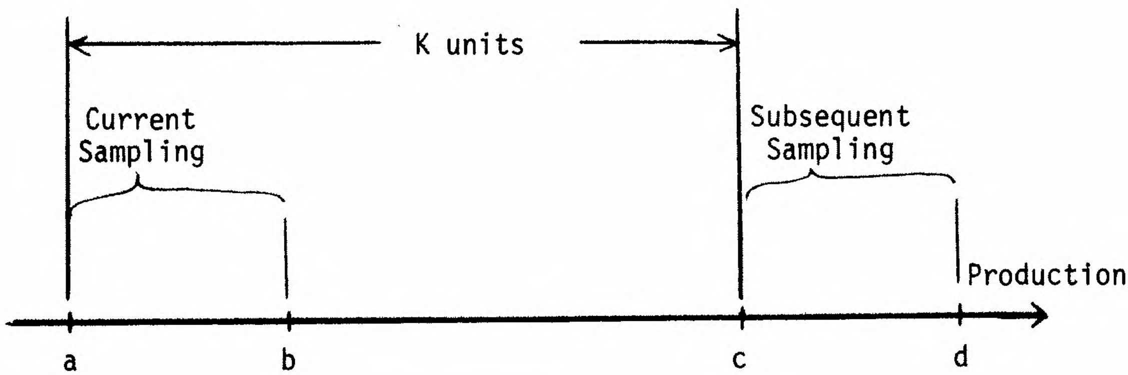

Figure 1. Definition of K.

2.2.2 Expected Cost of Investigating and Correcting the Process

The cost of investigating and/or correcting the process will be incurred whenever the value of the test statistic falls in the critical region. If y is a random variable which assumes the value one if the test statistic falls in the critical region, and the value zero if the test statistic does not fall in the critical region, and Z is the cost of investigating and correcting the process, then

$$C_2 = \frac{yZ}{K}$$

The random variable y can assume the value one under two conditions.

In the first condition, the process can in fact be out of control and this is detected by the control procedure. The probability of this outcome depends on the joint probability of the process being out of control at the time of the test and the probability that the test can detect this fact (usually called the power of the test). If δ_1 is the steady state probability of being out of control at the time of the test, β the long run probability of making a type II error, and $(1 - \beta)$ the power of the test, then the probability of being out of control and detecting it is

$$\delta_1(1 - \beta)$$

The second condition under which y can assume the value one occurs when the process is in control and a type I error is made. The probability of this outcome is dependent on the joint probability of the process being in control at the time of the test and the probability of making a type I error. If δ_0 is the steady state probability of being in control at the time of the test and α is the probability of making a type I error, then the probability is

$$\delta_0\alpha$$

Thus, the probability of y having a value of one is the sum of these two joint probabilities:

$$Pr(y = 1) = \delta_1(1 - \beta) + \delta_0\alpha$$

This quantity must be multiplied by the cost of investigating and/or correcting the process. Knappenberger and Grandage (10) base this cost on the generally available prior information concerning the number of times the process goes out of control, the length of time the process is stopped for repairs and the cost per hour (including repair costs) of an inoperative process. Rather than form an elaborate cost function, this information is used to establish an average cost, a_3. To obtain a cost per unit this must be divided by the number of units produced between successive samples. The expected cost of having to investigate and/or correct the process is the product of the cost per unit and the probability of investigating and correcting:

$$E(C_2) = \frac{a_3}{K} [\delta_1(1 - \beta) + \delta_0\alpha]$$

2.2.3 Expected Cost of Producing Defectives

When the process is in control there will be a small percentage of defectives produced due to chance variations in the output. If the process goes out of control and the shift is detected by the manufacturer, no additional defectives are produced because production is stopped. When the process goes out of control and the shift is undetected, a larger percentage of defectives may be produced. The buyer may react in several ways: he may reject just defects, or the entire lot; he might look for another supplier; and he might communicate his dissatisfaction to others. No matter what action is taken the end result is a cost to the manufacturer. Because the exact relationships between the number of defectives and the cost per unit of each defective is difficult to determine and would result in unnecessary complication in the model, we will assume a simple linear relationship. This is done by defining a_4 as the cost to the manufacturer for each defective unit produced. Its value is chosen to approximate the actual cost of producing a defective regardless of the state in which the process is operating.

If w is a random variable which assumes the value one if the unit is defective and the value zero otherwise, the expected cost per unit of producing defectives is

$$E(C_3) = a_4 Pr(w = 1)$$

If we let f_0 be the percentage of defectives produced when the process is in control, and let γ_0 be the probability that the process is in control at any point in time, then the probability that any unit produced is defective when the process is in control is

$$f_0 \gamma_0$$

Similarly, if f_1 is the percentage of defectives produced when the process is out of control and γ_1, is the probability that the process is out of control at any time, then the probability that any unit produced is defective while the process is out of control is

$$f_1 \gamma_1$$

Since the events described by the probabilities $f_0\gamma_0$ and $f_1\gamma_1$ are mutually exclusive, the total probability is the sum of the two joint probabilities:

$$Pr(w = 1) = f_o\gamma_o + f_1\gamma_1$$

The expected cost of producing defectives is the product of the cost per unit and the probability of producing defectives:

$$E(C_3) = a_4(f_o\gamma_o + f_1\gamma_1)$$

The total cost equation then becomes

$$E(C) = \frac{a_1 + a_2N}{K} + \frac{a_3}{K}[\delta_1(1-\beta) + \delta_o\alpha] + a_4(f_o\gamma_o + f_1\gamma_1)$$

2.3 Development of Probabilities

This section will present an explicit development for the probabilities δ_o, δ_1, γ_o, γ_1 and β in terms of system parameters.

2.3.1 Development of the Probabilities δ_0 and δ_1

As defined earlier, δ_o is the probability that the process is in control at the time of the test and δ_1 is the probability that the process is out of control at the time of the test. In Figure 1 the time of the test is defined to be at point a, where the current sampling starts. To develop δ_o and δ_1, a transition probability matrix ($\underline{B}$) is required. The elements, b_{ij}, of this matrix are the probabilities of moving from state i to state j during the production of K units. The definition of a transition matrix requires that

$$\sum_{j=o}^{s} b_{ij} = 1, \quad \text{for all } i$$

and $$0 < b_{ij} < 1, \text{ for all } i \text{ and } j$$

In order to define the probabilities b_{ij} it is necessary to determine the probabilities p_o and p_1, where p_o is the probability that the process will remain in control during the period, and p_1 is the probability that the process shifts out of control during the period.

If we assume that the time the process remains in control is an exponential random variable with mean λ^{-1} hours, then the probability of remaining in control for h hours is

$$p_o = 1 - \int_o^h e^{-\lambda t} dt = e^{-\lambda h}$$

If the production rate is R units per hour and if the production of fractional units is allowed, then the number of hours needed to produce K units is

$$h = \frac{K}{R}$$

and

$$p_o = e^{-\lambda k/R}$$

Since λ^{-1} is defined as the mean time in hours before a shift occurs, then the average number of units produced before a shift occurs is

$$\lambda' = \lambda/R$$

Then p_o becomes

$$p_o = e^{-\lambda' K}$$

and p_1 becomes

$$p_1 = 1 - e^{-\lambda' K}$$

Now consider the transition probabilities b_{ij}, where i is the initial state and j is the final state. Their definition requires that we assume that when the process goes out of control, it will stay out of control until the shift is detected (until the value of the test statistic falls in the critical region). That is, the process will not correct itself. Also we must assume that no shifts can occur while current sampling is taking place. In Figure 1, this is from point a to point b.

The probabilities b_{ij} can be assembled into a transition matrix $\underline{B}$ as follows:

$$\underline{B} = \begin{bmatrix} b_{00} & b_{01} \\ b_{10} & b_{11} \end{bmatrix}$$

First consider the probability b_{00}. This allows for the case in which the process starts the period in control and ends the period in control. The only way this can occur is to have no shift, and

$$b_{00} = p_0$$

The probability b_{01} allows for the case in which the process starts the period in control and ends the period out of control. This can occur only if there is a shift during the period, and hence

$$b_{01} = p_1$$

The probability b_{10} allows for the case in which the process starts the period out of control and ends the period in control. This can only occur if the out of control condition is detected and then no shift occurs for the remainder of the period, and

$$b_{10} = (1 - \beta)p_o$$

The probability b_{11} allows for the case in which the process starts the period out of control and ends the period out of control. There are only two ways this can occur. First, the out of control condition can go undetected, in which case it will continue in that state for the remainder of the period. Second, the out of control condition can be detected, but the process later shifts out of control again during the remainder of the period. If β is the long run probability of making a type II error at any time, then b_{11} is

$$b_{11} = \beta + (1 - \beta)p_1$$

The transition matrix $\underline{B}$ can now be written

$$\underline{B} = \begin{bmatrix} p_0 & p_1 \\ (1-\beta)p_0 & \beta + (1-\beta)p_1 \end{bmatrix}$$

$\underline{B}$ is the transition matrix of an irreducible, aperiodic, positive recurrent Markov chain. Therefore there exists a vector $\underline{\delta} = (\delta_0, \delta_1)$ such that

$$\underline{\delta}\ \underline{B} = \underline{\delta}$$

If we substitute the terms of $\underline{B}$ and of $\underline{\delta}$ into the above equation we have

$$\begin{bmatrix} \delta_0 & \delta_1 \end{bmatrix} \begin{bmatrix} p_0 & p_1 \\ (1-\beta)p_0 & \beta+(1-\beta)p_1 \end{bmatrix} = \begin{bmatrix} \delta_0 \\ \delta_1 \end{bmatrix}$$

Multiplying yields the following:

$$\delta_0 p_0 + \delta_1(1-\beta)p_0 = \delta_0$$

$$\delta_0 p_1 + \delta_1(\beta + (1-\beta)p_1) = \delta_1$$

These two equations are dependent since they both reduce to the same equality after substituting $p_1 = 1 - p_0$ and $\delta_1 = 1 - \delta_0$. That is, both equations reduce

$$\delta_0 = \frac{p_0(\beta - 1)}{\beta p_0 - 1}$$

This equation can be rewritten to give β as a function of δ_0 and p_0. Using the above expressions, β becomes

$$\beta = \frac{-1 + \delta_1 + p_0}{p_0 \delta_1}$$

2.3.2 Development of the Probabilities γ_0 and γ_1

As defined earlier γ_0 is the steady state probability of the process being in control at any point in time and γ_1, is the steady state probability of the process being out of control at any point in time. Since the process is either in control or out of control at any point in time, $\gamma_0 + \gamma_1 = 1$.

It was previously assumed that when the process went out of control, it would stay out of control until the shift was detected. Given this assumption, it is possible to define γ_0 as the sum of two conditional probabilities. The first of these describes the event in which the process starts the period in control and does not shift during the entire production of K units. This probability can be written

$$\varepsilon_0 p_0$$

where ε_0 is the steady state probability of starting in control at the end of current sampling (in other words, at point b in Figure 1) and p_0 is the probability of staying in control during the production of K units (recall that we assume no shifts can occur during sampling).

The second conditional probability describes the event in which the process starts the period in control and shifts out of control at some point in time after the start of the period. The probability can be written:

$$\varepsilon_0 F p_1$$

where p_1 is the probability that the process shifts from the in control to the out of control states and F is the average fraction of the production period that elapses before a shift occurs.

The steady state probability of being in control at any point in time (γ_0) can now be written:

$$\gamma_0 = \varepsilon_0 p_0 + \varepsilon_0 F p_1$$

Since $p_1 = 1 - p_0$, this can be rewritten as

$$\gamma_0 = \varepsilon_0 p_0 + \varepsilon_0 F(1 - p_0)$$

Duncan (4) has shown that, given the time before a process shifts out of control is an exponential random variable with mean λ^{-1} hours, the average time elapsed during the h hour interval between the i^{th} and $(i+1)^{st}$ samples before a shift occurs is

$$\bar{h} = \frac{\int_{ih}^{(i+1)h} e^{- \ t}(t-ih)dt}{\int_{ih}^{(i+1)h} e^{-\lambda t}\lambda dt}$$

$$\bar{h} = \frac{e^{-\lambda ih}\int_{0}^{h} e^{-\lambda t}\lambda t dt}{e^{-\lambda ih}\int_{0}^{h} e^{-\lambda t}\lambda dt}$$

$$\bar{h} = \frac{1 - (1 + \lambda h)e^{-\lambda h}}{\lambda(1 - e^{-\lambda h})}$$

where h is the number of hours to produce K units. To find the average fraction of the production interval before a shift occurs, we divide $\overline{h}$ by h:

$$F = \frac{\overline{h}}{h} = \frac{1 - (1 + \lambda h)e^{-\lambda h}}{\lambda h(1 - e^{-\lambda h})}$$

If R is the production rate, then F can be expressed in terms of units as

$$F = \frac{1 - (1 + \frac{\lambda K}{R})e^{-\lambda K/R}}{\frac{\lambda K}{R}(1 - e^{-\lambda K/R})}$$

Note the probability of starting the period in control, (ε_0) can be written in terms of other probabilities. The steady state probability of being in control at the time of the test (δ_0) is equal to the probability of starting the period in control and not shifting during the production of K units. That is,

$$\delta_0 = \varepsilon_0 p_0$$

Rearranging terms,

$$\varepsilon_0 = \frac{\delta_0}{p_0}$$

After this substitution is made, the probability that the process is in control at any point in time (γ_0) becomes

$$\gamma_0 = \underbrace{\delta_0 + \frac{F\delta_0}{p_0}(1 - p_0)}_{\text{Program use this expression ; correct}} = \delta_0\left[1 + \frac{F}{p_0}(1 - p_0)\right]$$

2.3.3 Development of the Probability β

As defined earlier, β is the long run probability of making a type II error. In order to develop an expression for β, it is necessary to define two other probabilities. These are $\underline{P}_i$, the conditional probability of making a type II error, given that i samples in the test statistic are from the out of control population, and p(i), the probability that i samples in the test statistic came from the out of control population. $\underline{P}_i$ is the probability of type II error on a test in which the test statistic (T_t) falls within the control limits when the process has been out of control for i periods. The process may have gone out of control in the most recent period (in which case there will be one sample in the test statistic from an out of control population), in the second most recent period (two samples from an out of control population), third most recent period (three samples), up to the M^{th} most recent period (M samples). If the process went out of control more than M periods ago, the test statistic will still contain M samples from an out of control state. If LCL and UCL are the lower and upper control limits, respectively, then $\underline{P}_i$ can be written:

$$\underline{P}_i = \Pr(LCL \leq T_t \leq UCL/i \text{ samples from out of control process})$$

At this point it may be helpful to review how the test statistic (T_t) is calculated. The first step is to take a sample of individual observations (X_{ti}) of size N and compute the statistic $\overline{X}_t$:

$$\overline{X}_t = \frac{1}{N} \sum_{i=1}^{N} X_{ti}$$

If we assume that the $\overline{X}_{ti}$ are normally distributed with mean μ and variance σ^2, then $\overline{X}_t$ will be distributed

$$\overline{X}_t \sim N(\mu, \sigma^2/N)$$

The second step is to use $\overline{X}_t$ to compute T_t:

$$T_t = \frac{1}{m}(\overline{X}_{t-m+1} + \overline{X}_{t-m+2} + \ldots + \overline{X}_t)$$

T_t will be normally distributed with mean μ_T and variance σ_T^2. The mean μ_T is a function of the number of terms in the test statistic that come from an in control process and the number that come from an out of control process. If μ_0 is the value of the in control process mean, μ_1 the value of the out of control process mean, and i the number of out of control periods, then

$$\mu_T = \frac{(M-i)\mu_0 + i\mu_1}{M}$$

If i = 0 (no terms from an out of control population), this expression yields

$$\mu_T = \mu_0$$

If i = M (all terms from an out of control population), this expression yields

$$\mu_T = \mu_1$$

The variance σ_T^2 is a function of the sample size (N) and the moving average index (M). If σ^2 is the variance of the individual X_{ti} then σ_T^2 can be written as

$$\sigma_T^2 = \frac{\sigma^2}{MN}$$

If we standardize T_t by subtracting the mean and dividing by the standard deviation, the T becomes T' where

$$T' = \frac{T - \mu_0}{\sigma/\sqrt{MN}}$$

If the process is in control, then T' is distributed normally with a mean of zero and a variance of one. T' can be used in conjunction with standard normal tables to evaluate the probability of being in control for different values of the critical region parameter. If α is the critical region parameter, and $Z\alpha/2$ is the ordinate of the standard normal distribution, then the probability that the test statistic falls outside the critical region is

$$\Pr(-Z\alpha/2 < T' < Z\alpha/2)$$

If the process has been out of control for the last i periods, then T is distributed

$$T \sim N\left(\frac{(m-i)\mu_0 + i\mu_1}{M}, \frac{\sigma^2}{MN}\right)$$

and T' is distributed

$$T' \sim N\left(\frac{\frac{(M-i)\mu_0 + i\mu_1}{M} - \mu_0}{\sigma/\sqrt{MN}} ,\ 1 \right)$$

Now the probability that the test statistic will fall outside the critical region with i samples from the out of control population is $\underline{P}_i$ where

$$\underline{P}_i = Pr(-Z\alpha/2 < T' < Z\alpha/2)$$

To standardize T', we subtract K_i where K_i is defined by

$$K_i = \frac{\frac{(M-i)\mu_0 + i\mu_1}{M} - \mu_0}{\sigma/\sqrt{MN}}$$

P_i is now

$$\underline{P}_i = Pr(-Z\alpha/2 - K_i \leq Z \leq Z\alpha/2 - K_i)$$

Where Z is distributed normally with mean of zero and variance of one. $\underline{P}_i$ can now be found through the use of a standard normal table.

The next probability to be developed is p(i), the probability that i samples in the test statistic came from the out of control population. Consider first p(1), the probability that one sample in the test statistic came from the out of control population. This means that (M-1) samples came from the in control population. We have previously assumed that when the process goes out of control it stays out of control. For one sample to have come from the most recent period, it must have come from the most recent period. Thus P(1) will be equal to the probability that the process started in control, m periods ago (ε_0), times the probability of running m-1 periods without having a shift (p_0^{M-i}) times the probability that a shift occurs in the most recent (M^{th}) period (p_1). Thus p(1) can be written

$$p(1) = \varepsilon_0 p_0^{M-i} p_1$$

Now consider p(2), the probability that two samples came from the out of control population, and M-2 samples from the in control population; p(2) will be equal to the probability that the process started in control, M periods ago (ε_0), times the probability of running M-2 periods without having a shift (p_0^{M-2}) times the probability that a shift occurs in the second most recent period (p_1) times the probability of failing to detect the shift in the most recent period (β). Thus p(2) can be written

$$p(2) = \varepsilon_0 p_0^{M-2} p_1 \beta$$

Similarly, p(3) can be written

$$p(3) = \varepsilon_0 p_0^{M-3} p_1 \beta^2$$

and p(i), the probability of having i samples from the out of control population in the test statistic can be written

$$p(i) = \varepsilon_0 p_0^{M-i} p_1 \beta^{i-1}$$

where

$$i = 1,2,\ldots,M-1$$

For the case where all M samples came from the out of control population, there are two possibilities. First, the shift can occur in the initial period. Then p(M) will be equal to the probability of starting in control M periods ago (ε_0), times the probability that the process shifts out of control in the first period (p_1) times the probability that the shift goes undetected for the remaining M-1 periods (β^{M-1}). The probability can be written

$$\varepsilon_0 p_1 \beta^{M-1}$$

The second possibility allows for the case in which the process was out of control at the start of the first period. If we define ε_1 to be the probability that the process starts out of control (note that $\varepsilon_1 = 1 - \varepsilon_0$), and the probability that the out of control condition goes undetected for the remaining M-1 periods, then the probability can be written

$$\varepsilon_1 \beta^{M-1}$$

The probability that all M samples came from an out of control population is the sum of the probabilities of these two cases:

$$p(M) = \varepsilon_0 p_1 \beta^{M-1} + \varepsilon_1 \beta^{M-1}$$

Now it is possible to calculate β, the long run probability of making a type II error. According to the total probability law, β is equal to the product of the conditional probability of making a type II error, given that i samples are from the out of control population, and the probability that i samples are from the out of control population, summed over all i:

$$\beta = \sum_{i=1}^{M} \underline{P}_i p(i)$$

$$= \sum_{i=1}^{M} \underline{P}_i \varepsilon_0 p_0^{M-i} p_1 \beta^{i-1} + \underline{P}_M \varepsilon_1 \beta^{M-1}$$

$$= \sum_{i=1}^{M} Pr(-Z\alpha/2 - K_i \leq Z \leq Z\alpha/2 - K_i)\varepsilon_0 p_0^{M-i} p_1 \beta^{i-1}$$

$$+ Pr(-Z\alpha/2 - K_i \leq Z \leq Z\alpha/2 - K_i)\varepsilon_1 \beta^{M-1}$$

Solving this expression is simplified if we make use of the following previously developed equations:

$$\varepsilon_0 = \frac{\delta_0}{p_0}$$

$$\varepsilon_1 = 1 - \varepsilon_0 = 1 - \frac{\delta_0}{p_0}$$

$$p_1 = 1 - p_0$$

$$\beta = \frac{-1 + \delta_1 + p_0}{p_0 \delta_1}$$

Substituting these terms wherever possible results in the following equation.

$$\frac{-1+\delta_1+p_0}{p_0\delta_1} = \delta_0 \sum_{i=1}^{M} P_i p_0^{M-i-1} p_1 \left(\frac{-1+\delta_1+p_0}{p_0\delta_1}\right)^{i-1} + P_M\left(1-\frac{\delta_0}{p_0}\right)\left(\frac{-1+\delta_1+p_0}{p_0\delta_1}\right)^{M-1}$$

Simplifying, we have

$$\delta_0 = -(1-\delta_0)\delta_0 \sum_{i=1}^{M} P_i p_0^{M-i} p_1 \left(\frac{p_0-\delta_0}{p_0(1-\delta_0)}\right)^{i-1} - p_0(1-\delta_0)P_M\left(1-\frac{\delta_0}{p_0}\right)\left(\frac{p_0-\delta_0}{p_0(1-\delta_0)}\right) + p_0$$

Numerical methods for the solution of this equation and the evaluation of the cost function are presented in the next chapter.

2.4 Probability of Startup Operation

In an earlier section, moving average control procedures were characterized as having two states of operation: a startup state, which lasts from the initial period of operation up to period M-1, and a steady state, which lasts from period M until the process is stopped for investigation and correction. In the model that was developed it was assumed that the procedure was operating in the steady state. Now that all the probabilities have been defined, it is possible to discuss this assumption. The probability that a period will be a startup period can be approximated as follows. If γ_1 is the probability of being out of control at any point in time, $(1-\beta)$ the probability of being able to detect an out of control condition, γ_0 the probability of being in control at any point in time, then the probability that a period is a startup period is

$$\Pr(\text{Startup}) = \gamma_1(1-\beta) + \gamma_0\alpha$$

This is an approximation because it assumes that the procedure is starting up from steady state operation, rather than from startup operation. This probability has not been incorporated in the model because its influence on the expected cost is insignificant compared to the additional complexity it would add to the model. The effects of ignoring the startup probability cannot be examined until numerical results have been obtained. This is accomplished in the next chapter.

CHAPTER III

SOLUTION METHOD AND NUMERICAL RESULTS

The purpose of this chapter is to formulate a method for selecting the sample size (N), the interval between samples (K), the critical region parameter (α), and the moving average index (M), which minimize the value of the expected cost function (E(C)). In addition, an example of an application of the model developed in the previous chapter will be presented, and the performance of the moving average procedure under a variety of parameter configurations will be reported.

3.1 Optimization Techniques

The first optimization technique to be considered was to construct implicit functions for the partial derivatives of the expected cost with respect to M N, K and α. This method was rejected because these partial derivatives are not independent (they are all functions of the transition matrix $\underline{B}$) and because of the complexity of the equation for the steady state probability of being in control on any test (δ_0).

Another method which was considered was to use a simultaneous search technique to evaluate the expected cost function across a grid covering the cost surface. Duncan (4) used this technique to obtain information about the nature of the cost function developed in his paper. His work indicated that cost functions that arose from economic models of control charts were likely to have a number of local minima. Because of this, a simultaneous search technique seemed most effective.

The employment of this technique requires the use of a digital computer. Accordingly a computer program was written, and the flow chart is shown in Figure 2. The complete listing of the program is given in the appendix, along with a translation of the computer symbols into the symbols used in the model described in Chapter II. The next section illustrates the use of the computer procedure through the use of a numerical example.

3.2 A Numerical Example

The following costs and parameter values (M, N, K and α) of this example were chosen arbitrarily.

a_1 = \$1	M = 3	$\mu_0 = 10$
a_2 = \$1	N = 1	$\sigma = 3$
a_3 = \$10	K = 40	$\lambda' = 1000$
a_4 = \$10	$\alpha = .002$	

The value of μ_1 that was used (17.2) derives from defining the shift to be 2.4σ. The value of f_0 (.0027) is that fraction of the output of an in control process whose mean lies outside the interval $\mu_0 \pm 3\sigma$, and the value of f_1 (.2742) is that fraction of the output of an out of control process whose mean lies outside the interval $\mu_0 \pm 3\sigma$.

The first quantities to be calculated are the probability that no shift occurs during the production of K units (p_0), and the probability that a shift does occur during the production of K units (p_i).

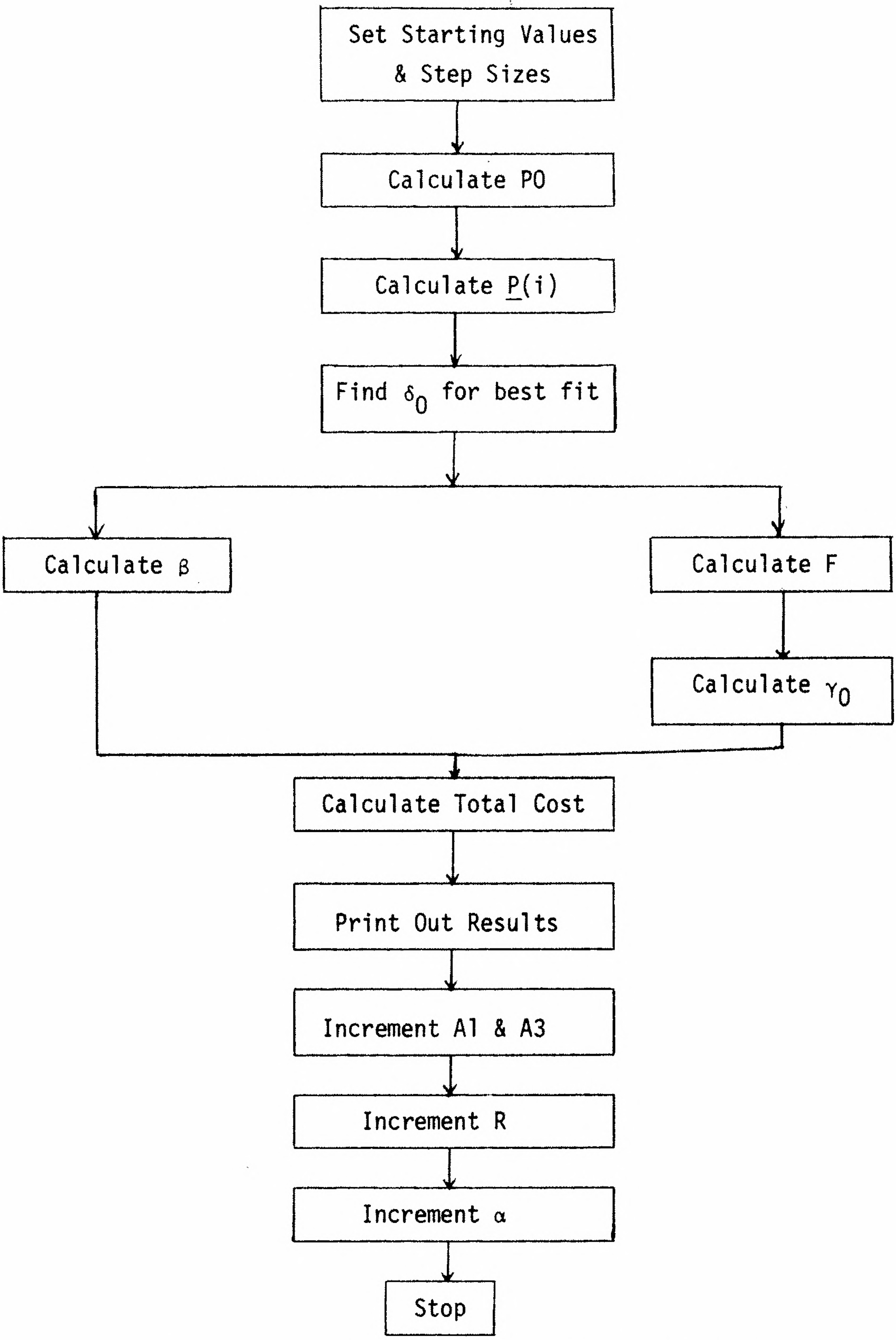

Figure 2. Flow Diagram of Numerical Procedure

$$p_0 = e^{-\lambda K/R} = .96$$

$$p_1 = 1 - p_0 = .04$$

The procedure now calculates the conditional probability of making a type II error on a test, given that i samples in the test statistic are from the out of control population ($\underline{P}_i$):

$$\underline{P}_i = \Pr(-Z\alpha/2 - K_i \leq Z \leq Z\alpha/2 - K_i),\ i = 0,1,2,\ldots,M$$

<u>M=1</u>

$\underline{P}_0 = .9979$
$\underline{P}_1 = .7549$

<u>M=2</u>

$\underline{P}_0 = .9979$
$\underline{P}_1 = .9181$
$\underline{P}_2 = .3805$

<u>M=3</u>

$\underline{P}_0 = .9979$
$\underline{P}_1 = .9558$
$\underline{P}_2 = .6205$
$\underline{P}_3 = .1430$

Having calculated p_0, p_1, and $\underline{P}_i$, it is now possible to calculate the steady state probability of being in control on any test (δ_0).

Recall that δ_0 could be expressed as

$$\delta_0 = -\delta_0 \sum_{i=1}^{M} \underline{P}_i p_0^{M-i}\, p_1 \left(\frac{p_0 - \delta_0}{p_0(1-\delta_0)}\right)^{i-1} - p_0 \underline{P}_M \left(1 - \frac{\delta_0}{p_0}\right)\left(\frac{p_0 - \delta_0}{p_0(1 - \delta_0)}\right)^{M-1} + p_0$$

Note that this expression is a polynomial in δ_0 of order M, which means that up to M different roots may exist. In order to explore this possibility, the expression was solved algebraically for $M = 2$ and $M = 3$; in each case only one positive root was found. The expression was evaluated for larger values of M during initial computer runs, and a similar result was found. An examination of the above equation reveals that δ_0 cannot be larger than p_0, or smaller than zero (else the expression

$$\frac{p_0 - \delta_0}{p_0(1 - \delta_0)}$$

which appears in the above equation will be negative). Accordingly, this range of possible values is divided into a number of equal size increments (say .01 each) and the value of δ_0 at each increment (0.0, 0.01, 0.02, etc.) is then used as a trial value in the above equation. The "goodness of fit" of each trial value is established by computing the absolute value of the difference in the left hand side and right hand side of the above equation. The value of δ_0 which gives the best fit is selected and the corresponding right hand side is used as δ_0 in the economic model. For this example, $\delta_0 = .96$ gives the best fit and the corresponding right hand side is .9597.

Having calculated the steady state probability of being in control on any test, it is possible to calculate the probability of making a type II error on a test (β).

$$\beta = \frac{p_0 - \delta_0}{p_0(1 - \delta_0)} = \frac{.96 - .9597}{.96(1 - .9597)} = .0075$$

It is also possible to calculate the average fraction of an interval that elapses before a shift occurs (F) and the probability of being in control at any point in time (γ_0).

$$F = \frac{1 - (1 + \lambda K/R)e^{-\lambda K/R}}{\lambda K/R(1 - e^{-\lambda K/R})} = .496$$

$$\gamma_0 = \delta_0 + F(\frac{\delta_0}{p_0})(1 - p_0) = .979$$

Finally, the total cost function itself is calculated

$$E(C) = \frac{a_1 + a_2 N}{K} + \frac{a_3}{K}[\delta_1(1 - \beta) + \delta_0 \alpha] + a_4(f_0\gamma_0 + f_1\gamma_1)$$

$\underline{M = 1}$
$E(C) = .137$

$\underline{M = 2}$
$E(C) = .136$

$\underline{M = 3}$
$E(C) = .135$

3.3 Experimental Results

The purpose of this section is to present the cost structure of the moving average control chart given several values of the cost coefficients and the critical region parameter, and four values of the size of the shift (.5σ, 2.4σ, 3.6σ and 4.8σ). All results in this section are for a mean deterioration rate (λ') of 1000. The effect of changing this parameter is discussed in a later section.

Consider first the case of a shift size of .5σ, shown in Table 1. A shift this small means that less than one percent of the output

Table 1. Optimal Values of M, N & K for Moving Average Chart with Shifts of .5σ and λ' = 1000.

a_1	a_2	a_3	a_4	α	M	N	K	$	
1	1	10	10	0001	5	1	280	.049	4,1,320,360,400,440,5,1,480-560,6,1,600
				001	5	1	280	.049	4,1,320-440,5,1,480-560,6,1,600
				002	5	1	280	.049	4,1,320-440,5,1,480-560,6,1,600
				005	5	1	280	.049	4,1,320-440,5,1,480-560,6,1,600
				01	6	1	280	.049	5,1,320,4,1,360-440,5,1,480-560,6,1,600
				02	5	1	320	.049	360,4,1,400,5,1,440-560,6,1,600
				05	6	1	360	.049	400,4,1,440,6,1,480-520
				.1	6	1	360	.050	5,1,400,4,1,440-520,5,1,560-640,6,1,680
1	1	100	10	0001	1	1	760	.075	800-1000
				001	1	1	760	.075	800-1000
				002	1	1	760	.075	800-1000
				005	1	1	840	.075	880-1000
				01	1	1	1000	.075	
				02	1	1	960	.077	1000
				05	1	1	1000	.080	
				.1	1	1	1000	.085	
10	1	10	10	0001	5	1	880	.061	920-960,4,1,1000
				001	5	1	880	.061	920-960,4,1,1000
				002	5	1	880	.061	920-960,4,1,1000
				005	5	1	880	.061	920-960,4,1,1000
				01	5	1	880	.061	5,1,920-960,4,1,1000
				02	6	1	880	.061	5,1,920-960,4,1,1000
				05	5	1	920	.061	960,4,1,1000
				.1	5	1	960	.061	1000
10	1	100	10	0001	1	1	960	.084	1000
				001	1	1	960	.084	1000
				002	1	1	960	.084	1000
				005	1	1	960	.084	1000
				01	1	1	1000	.084	
				02	1	1	960	.086	1000
				05	1	1	1000	.089	
				.1	1	1	1000	.094	

produced when the process is out of control will lie outside $\mu_0 \pm 3\sigma$. Under these conditions, it will not be feasible to sample very often since the cost of bad production is so much less than the cost of sampling. This can be seen in Table 1. Even when the fixed cost of sampling (a_1) is small, the sampling interval lies in the range 280-360 units. When a_1 is larger, the sampling interval approaches 1000 units. Increasing a_1 does not seem to affect the size of the moving average index (M) or the sample size (N), which is always one.

The effects of changing the cost of searching and correcting (a_3) are somewhat different. When a_1 is small, increasing a_3 causes the sampling interval to more than double, while the size of the moving average index decreases to the smallest possible value. Apparently the cost of searching and correcting is so high that the risk of incurring it must be minimized by not sampling except at intervals of 760-1000 units. When a_1 is large, increasing a_3 causes only a slight increase in K, which is already at a high level, while the size of M is decreased as before.

The effect of increasing the critical region parameter (α) is negligible, for the range of values considered. This is a reflection of the small shift size.

The entries to the right of the cost column in Table 1 indicate alternative combinations of M, N and K that give the same (minimum) cost as that shown. For example, consider the case of $a_1 = 1$, $a_2 = 1$, $a_3 = 10$, $a_4 = 10$ and $\alpha = .005$. All of the following combinations give the same cost (\$.049):

M	N	K
5	1	280
4	1	320
4	1	360
4	1	400
4	1	440
5	1	480
5	1	520
5	1	560
6	1	600

This table is graphed in Figure 3.

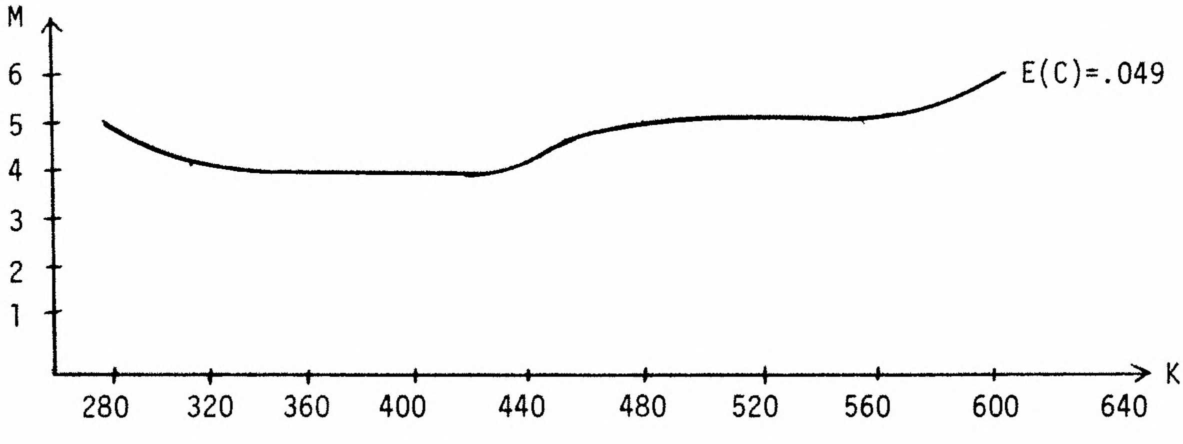

Figure 3. E(C) vs. M and K for the case of .5σ Shift, a_1=1, a_2=1, a_3=10, a_4=10, λ'=1000 and α =.005.

The number of these alternative values when a_3 is small suggests that increasing a_1 causes a more "peaky" cost surface.

Now consider the case of a shift size of 2.4σ, shown in Table 2. The sampling interval is reduced by a factor of seven to ten while costs are increased by a factor of three or four. The optimal sample size remains one in every case. The size of the moving average index (M) decreases for larger values of α when a_3 is small. When a_3 is large,

Table 2. Optimal Values of M, N, & K for Moving Average Chart with Shifts of 2.4σ and λ' = 1000.

a_1	a_2	a_3	a_4	α	M	N	K	$	
1	1	10	10	0001	6	1	40	.140	
				001	4	1	35	.141	3,1,40
				002	6	1	35	.141	3,1,40
				005	4	1	35	.142	3,1,40
				01	3	1	40	.143	
				02	3	1	40	.145	
				05	3	1	40	.152	
				.1	2	1	45	.163	
1	1	100	10	0001	4	1	40	.229	
				001	4	1	40	.229	
				002	5	1	40	.233	
				005	4	1	45	.240	
				01	4	1	50	.250	
				02	2	1	50	.269	
				05	3	1	75	.311	
				.1	3	1	95	.364	
10	1	10	10	0001	5	1	90	.278	6,1,95,100
				001	5	1	90	.278	6,1,85,5,1,95
				002	4	1	90	.278	6,1,85,4,1,95,5,1,100
				005	4	1	90	.278	4,1,95,5,1,100
				01	5	1	90	.278	5,1,95
				02	5	1	90	.279	4,1,95
				05	4	1	90	.282	3,1,95
				.1	3	1	95	.287	100
10	1	100	10	0001	5	1	95	.364	6,1,100
				001	5	1	95	.364	
				002	5	1	95	.365	6,1,100
				005	4	1	95	.368	4,1,100
				01	4	1	95	.373	4,1,100,4,1,105
				02	4	1	95	.382	3,1,100,3,1,105,4,1,110
				05	4	1	110	.406	3,1,115,4,1,120
				.1	4	1	125	.442	3,1,130,135,4,1,140

M no longer has a value of one but is rather two to five. In addition, the number of alternative parameter combinations which have the same cost as the minimum shown is reduced when a_1 and a_3 are small and increased when a_1 and a_3 are large. All of these effects are a reflection of the fact that a shift of 2.4 means about twenty-five percent of the output from an out of control process will lie outside $\mu_0 \pm 3\sigma$ limits.

The effect of increasing the fixed cost of sampling (a_1) is to approximately double the sampling interval and the costs when the cost of investigating and correcting (a_3) is small; when a_3 is large, sampling intervals and costs are increased by a factor of 1.5 to 2. The moving average index is changed slightly, but there is no definite pattern.

The effect of increasing a_3 is to increase the sampling interval for larger values of the critical region parameter and to slightly decrease most values of M. A larger critical region reinforces the effect of larger a_3 and the model compensates by reducing the sampling frequency.

The effect of increasing the critical region parameter (α) is to decrease the frequency of sampling, with the greatest effect being observed when a_1 is small and a_3 is large, and the smallest effect when a_1 is large and a_3 is small. This is due to the reinforcing effect of α and a_3 mentioned above. Increasing α causes a decrease in M, with the greatest effect occurring when a_3 is small. When a_3 is small, the effect of the cost of bad production (a_4) is proportionally greater; one way to reduce this cost is to reduce M (because of the "masking" effect which a large number of terms in the moving average can cause).

Next consider the case of a shift size of 3.6σ, shown in Table 3. A shift of this size means that almost three fourths of the output of an out of control process will fall outside $\mu_1 \pm 3\sigma$ control limits. Note that the low and high levels of a_1 have been increased by a factor of ten (to allow for comparison with the work of Knappenberger and Grandage). The sampling interval is slightly increased when a_1 is small and almost trebled when a_1 is larger. The optimal sample size remains one in every case. The size of the moving average index (M) is reduced for most values of α and cost coefficients, with the greatest change coming when a_1 and a_3 are small. Here the effect of a_4 is proportionally greater and this cost can be reduced by smaller values of M. The cost surface becomes generally more sensitive to changes in parameter values as evidenced by the reduced number of alternative minimum cost combinations.

The effect of increasing the fixed cost of sampling (a_1) is to double or triple the size of the sampling interval for every value of a_3. The moving average index is somewhat changed, but there is no recognizable pattern. The effect of increasing a_3 is to slightly increase the sampling interval for larger values of α. The effect on M is slight. The effect of increasing α is to slightly increase the size of the sampling interval and slightly decrease the size of M, for every value of a_1 and a_3.

Finally consider the case of a shift size of 4.8σ, shown in Table 4. A shift of this size means that less than one percent of the output of

Table 3. Optimal Values of M, N, & K for Moving Average Chart with a Shift of 3.6σ and λ'=1000.

a_1	a_2	a_3	a_4	α	M	N	K	$	
10	1	10	10	0001	3	1	55	.432	
				001	3	1	55	.432	
				002	3	1	55	.432	
				005	2	1	55	.433	3,1,60
				01	2	1	55	.434	60
				02	3	1	55	.435	
				05	2	1	55	.441	60
				.1	2	1	60	.448	
10	1	100	10	0001	3	1	55	.520	3,1,60
				001	3	1	55	.521	
				002	3	1	55	.522	60
				005	3	1	55	.528	2,1,60
				01	2	1	60	.536	
				02	2	1	60	.551	
				05	2	1	65	.595	70
				.1	2	1	75	.658	80
100	1	10	10	0001	5	1	170	1.211	4,1,175-185
				001	3	1	175	1.211	180,4,1,185
				002	3	1	175	1.211	180
				005	3	1	175	1.211	180
				01	3	1	175	1.211	180
				02	3	1	170	1.212	2,1,175-180,3,1,185
				05	2	1	175	1.213	180
				.1	3	1	175	1.215	2,1,180
100	1	100	10	0001	4	1	175	1.293	180
				001	4	1	180	1.294	
				002	3	1	175	1.294	180
				005	3	1	180	1.295	
				01	3	1	175	1.298	180,185
				02	3	1	180	1.302	
				05	2	1	180	1.316	185
				.1	2	1	180	1.339	190-195

Table 4. Optimal Values of M, N & K for Moving Average Chart with a Shift of 4.8σ and λ'=1000.

a_1	a_2	a_3	a_4	α	M	N	K	$	
10	1	10	10	0001	2	1	50	.493	
				001	2	1	50	.493	
				002	2	1	50	.494	
				005	2	1	50	.494	
				01	2	1	50	.495	
				02	2	1	50	.497	
				05	2	1	50	.503	
				.1	2	1	50	.512	
10	1	100	10	0001	2	1	50	.581	
				001	2	1	50	.583	
				002	2	1	50	.585	
				005	2	1	50	.590	
				01	2	1	50	.600	
				02	2	1	55	.618	
				05	2	1	60	.669	
				.1	2	1	65	.743	70
100	1	10	10	0001	2	1	150	1.396	3,1,155
				001	2	1	150	1.396	155
				002	2	1	150	1.396	155
				005	2	1	150	1.396	155
				01	3	1	155	1.396	
				02	2	1	150	1.397	155
				05	2	1	150	1.399	155
				.1	2	1	155	1.401	
100	1	100	10	0001	3	1	155	1.479	
				001	2	1	150	1.480	155
				002	3	1	155	1.480	
				005	2	1	150	1.482	155
				01	2	1	150	1.485	155
				02	2	1	155	1.490	
				05	2	1	155	1.507	160
				.1	2	1	160	1.534	165

an out of control process will fall within $\mu_0 \pm 3\sigma$ control limits. When the process goes out of control, the cost of bad production is incurred with a vengence; the behavior of the model reflects this fact. The sampling interval has decreased slightly for all values of a_1 and a_3, more frequent sampling increases the probability of detecting a shift when it occurs. The optimal sample size is one in every case. The size of the moving average index has been reduced; this decreases the "masking" effect of terms from an in control population and increases the probability of catching a shift as soon as it occurs. The number of alternative parameter combinations which have the same minimum cost is reduced further, indicating a more "peaky" cost surface.

The effect of increasing a_1 is to treble the sampling interval for all values of a_3 and α. Increasing a_3 has the same effect as with the smaller shift sizes, a larger sampling interval for larger values of α. The effect of increasing the critical region parameter is negligible.

3.4 Sensitivity to Increasing the Mean Deterioration Rate (λ')

To illustrate the effect on the optimal solution of a change in the mean deterioration rate of the process, the value of λ' has been increased from 1000 units between shifts to 10,000 units. The results, shown in Table 5, are for a shift size of 2.4σ, and can be directly compared with Table 2. The result to be expected is an increase in the size of K. Since assignable errors are occurring at one-tenth the frequency, it will be economically feasible to produce more units between samples. This is in fact what happens. The sampling interval is

Table 5. Moving Average Chart with Shift of 2.4σ and λ'=10,000.

a_1	a_2	a_3	a_4	α	M	N	K	$	
1	1	10	10	0001	4	1	110	.061	3,1,120-130,4,1,140
				001	3	1	110	.061	120-140
				002	3	1	110	.061	120-130,4,1,140
				005	6	1	110	.061	3,1,120-130
				01	2	1	110	.062	120-140,3,1,150
				02	4	1	130	.062	
				05	2	1	120	.065	1,1,130-150,2,1,160
				.1	2	1	140	.068	150-160,3,1,170
1	1	100	10	0001	4	1	110	.070	3,1,120-130,4,1,140
				001	4	1	100	.071	2,1,110-140,3,1,150
				002	5	1	120	.071	4,1,130
				005	6	1	110	.074	2,1,120-130,1,1,140,2,1,150-160
				01	4	1	130	.077	2,1,140-150,3,1,160-170
				02	3	1	160	.083	2,1,170-180
				05	2	1	200	.098	240-260
				.1	2	1	280	.116	290-310,3,1,320
10	1	10	10	0001	5	1	260	.105	4,1,270-300,5,1,310
				001	4	1	260	.105	270,3,1,280-300,4,1,310
				002	4	1	260	.105	3,1,270-300,4,1,310
				005	5	1	260	.105	3,1,270-300,4,1,310
				01	4	1	270	.105	3,1,280-300,4,1,310
				02	2	1	260	.106	270-310,3,1,330
				05	2	1	260	.107	270-280,1,1,290-300,2,1,310-330,3,1,340
				.1	3	1	280	.108	2,1,290-310,3,1,320
10	1	100	10	0001	5	1	260	.114	270-300,5,1,310
				001	2	1	270	.114	280-310
				002	5	1	290	.114	
				005	5	1	290	.115	300
				01	3	1	280	.117	290-320,4,1,330
				02	4	1	290	.120	3,1,300-330
				05	3	1	320	.129	2,1,340-360,3,1,370-380
				.1	3	1	360	.142	2,1,370-400,3,1,440

doubled or tripled and costs are reduced from one-third to one-twentieth of their value with a smaller mean deterioration rate. The increase in λ' also causes a decrease in the sensitivity of the moving average chart: the number of alternate parameter combination which have the same minimum cost is increased, especially for the case where a_1 is smaller. The size of M does not seem to be affected.

3.5 Behavior of the Cost Surface

The purpose of this section is to study the behavior of the moving average cost surface as it responds to changes in sampling interval and sample size. This was done by calculating the values of the expected cost function and the individual cost components (cost sampling and inspecting $E(C_1)$), cost of searching and correcting ($E(C_2)$), and cost of lost production ($E(C_3)$)), for several values of N, K and M. The value of the critical region parameter and the mean deterioration rate were held constant at .0001 and 1000, respectively. These results are shown in Figures 4 and 5.

3.6 A Comparison with the $\overline{X}$ Chart

The purpose of this section is to compare the cost of using the moving average chart with the cost of using the $\overline{X}$ chart. Data on the $\overline{X}$ chart is taken from Knappenberger and Grandage (10). Several values of the cost coefficients (a_1, a_2, a_3, and a_4), three values of the size of the shift (2.4σ, 3.6σ and 4.8σ) and one value of the mean deterioration rate ($\lambda' = 1000$) were used. The comparison is shown in Table 6.

The results indicate that the moving average control chart is

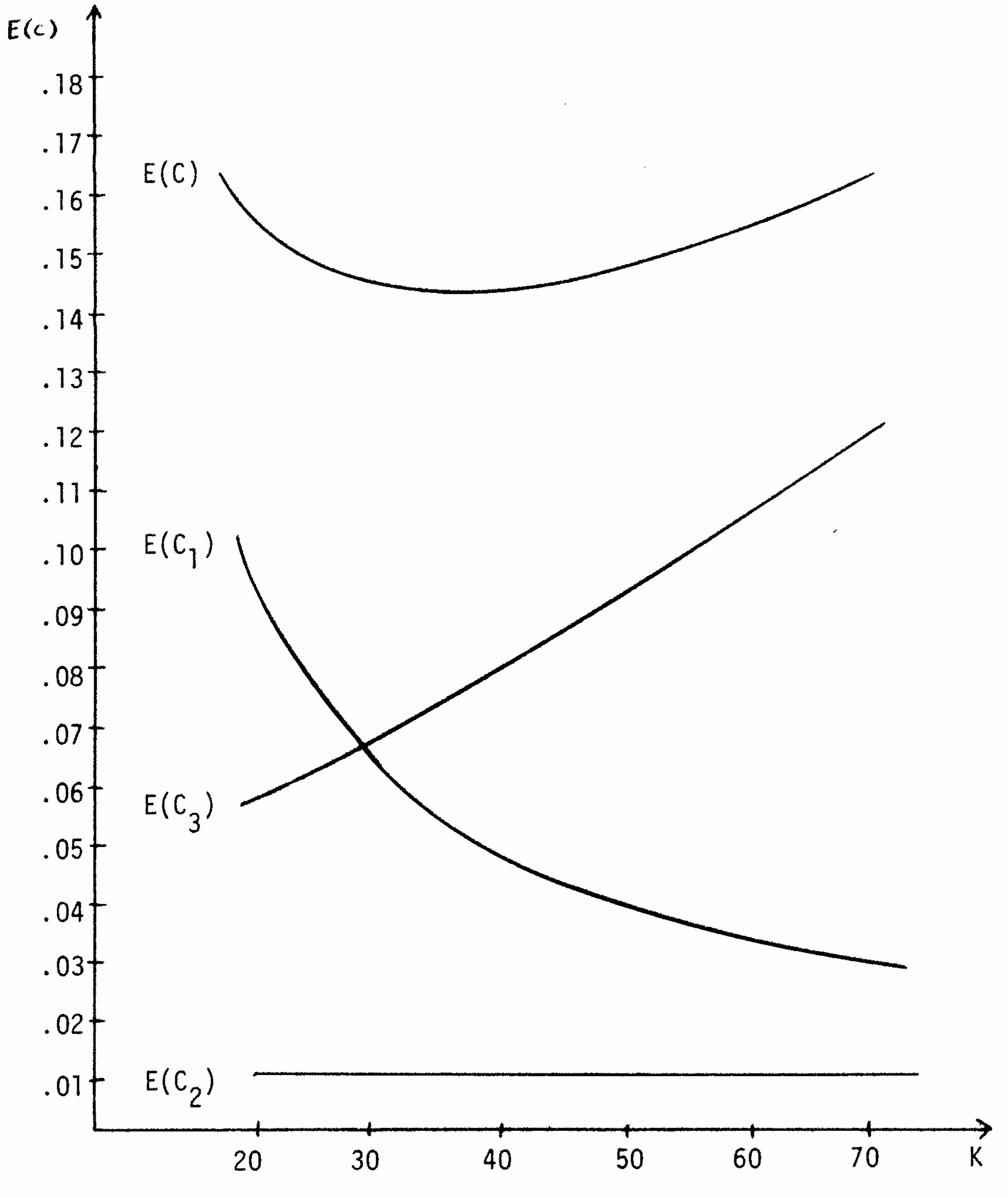

Figure 4. E(C) vs. K for the Case of N=1, M=3, $\lambda'=1000$, $a_1=1$, $a_2=1$, $a_3=10$, $a_4=10$.

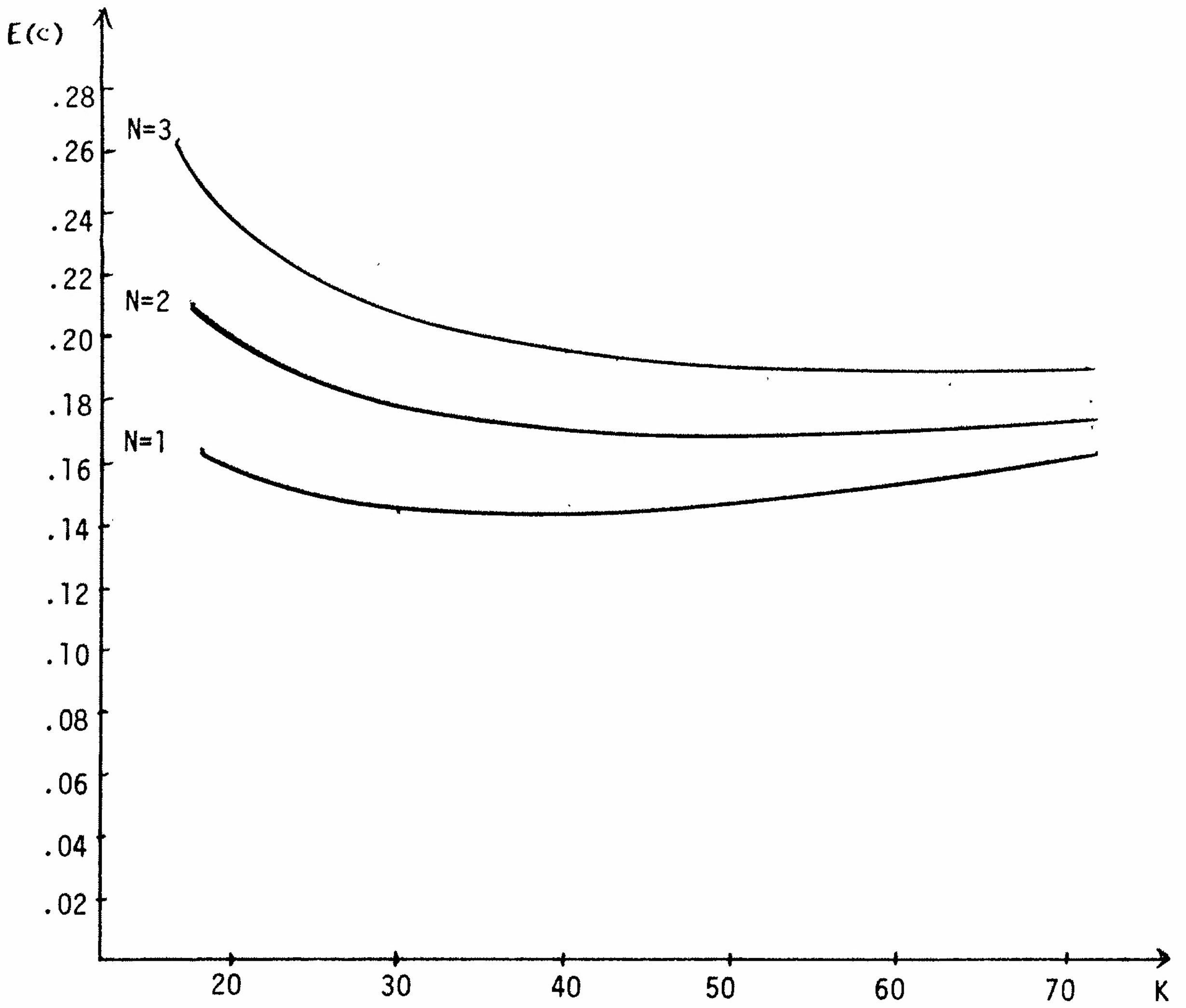

Figure 5. E(C) vs. K for the Case of N=1, 2, 3, M=3, $\lambda'=1000$, $a_1=1$, $a_2=1$, $a_3=10$, $a_4=10$

Table 6. Comparison Between the $\bar{X}$ Chart and the Moving Average Chart with $\lambda' = 1000$

					SHIFT=2.4σ		SHIFT=3.6σ		SHIFT=4.8σ	
					a_1		a_1		a_1	
a_4	a_2	a_3	Chart	Parameters	1.0	10.	10.	100.	10.	100.
10.	1.0	10.	M.A.	E(C)	.140	.278	.432	1.211	.493	1.396
				M	6	5	3	5	2	2
				N	1	1	1	1	1	1
				K	40	90	55	170	50	150
				α	.0001	.0001-.01	.0001-.002	.0001-.01	.0001-.001	.0001-.01
			$\bar{X}$	E(C)	.341	.610	.782	2.065	.846	2.335
				N	2*	2	2	3	2*	2
				K	22	46	26	105	30	90
				α	.0244	.080	.00046	.080	.006	.0244
		100	M.A.	E(C)	.229	.364	.520	1.293	.581	1.479
				M	4	5	3	4	2	3
				N	1	1	1	1	1	1
				K	40	95	55	175	50	155
				α	.0001	.0001-.001	.0001	.0001	.0001	.0001
			$\bar{X}$	E(C)	.460	.737	.875	2.205	.941	2.430
				N	2	3	2	3	2*	2
				K	20	46	34	85	30	85
				α	.0027	.006	.002	.00018	.00046	.00116

*Indicates that N=1 is optimal, but N=2 is used in order to allow for calculation of variance.

superior to the $\overline{X}$ control chart for all the values of the cost coefficients and shift sizes that were considered, the degree of advantage being dependent on the values of these parameters. The moving average chart consistently uses a larger sampling interval (K) in conjunction with a smaller sample size (N) than the $\overline{X}$ chart. Increasing K and decreasing N tends to make the fixed and variable costs of sampling and the cost of investigating and correcting smaller, at the expense of increased bad production. This means that the moving average chart has a smaller probability of type II error and a greater power to detect a shift. The optimal value of the critical region parameter for the moving average chart is consistently smaller than that for the $\overline{X}$ chart. This means that a moving average chart is comparatively sensitive to the cost of searching and correcting.

In order to understand the effects of changing the cost factors it is helpful to consider them each separately.

Consider first the effect of changing a_1, the fixed cost of sampling. The result expected from increasing a_1 is to decrease the frequency of sampling, until the cost of investigating and correcting and the cost of bad production become prohibitive. This is in fact what happens for both the $\overline{X}$ chart and the moving average chart. For a shift size of 2.4σ, increasing a_1 causes sampling frequency to be halved and costs doubled at both levels of a_3. For a shift of 3.6σ, the frequency of sampling is approximately a third as much and costs are three times as much when a_1 is increased. Similar results are obtained when the shift size is 4.8σ.

Increasing the size of a_1 does not seem to affect the optimal sample size (N) for either type of chart, but it does generally increase the size of the critical region parameter (α), and therefore the cost of inspecting and correcting.

The effect of increasing a_1 on the moving average parameter M seems to depend on the size of a_3. When a_3 is small, there is no consistent effect, but when a_3 is large, increasing a_1 causes an increase in the moving average parameter (M). This can be explained by the fact that a larger a_3 means it is less desirable to search and correct, and a larger M will insure this. As M increases, sensitivity to a shift out of control decreases, because of the masking effect of the "in control" terms in the moving average.

Now consider the effect of changing the value of a_3, the cost of investing and correcting the process. Increasing a_3 does not affect the frequency of sampling in the moving average chart for any size of shift considered. The sampling interval of the $\bar{X}$ chart in the case of 2.4σ and 4.8σ shifts does not change either, but when the shift size is 3.6σ, changes do occur. When a_1 is small, increasing a_3 causes an increase in K; when a_1 is large, increasing a_3 decreases K in a similar proportion. In other words, if fixed cost of sampling (a_1) is not significant, a larger cost of searching and correcting (a_3) can best be born by reducing the frequency with which you are susceptible to it, and vice-versa. Why this effect is observed only for 3.6σ shifts is not clear.

Increasing a_3 has the effect of decreasing the value of the critical region parameter for the $\overline{X}$ chart in every case. When a_3 is small, the moving average chart generally allows a range of values for the critical region parameter, but when a_3 is larger, only the smallest value is allowed (although when the shift is 2.4σ, this effect does not occur).

The effect of increasing a_3 on the moving average parameter (M) depends on the value of a_1 and the size of the shifts. M is decreased for a 2.4σ shift when a_1 is small and for a 3.6σ shift when a_1 is larger. M is increased for a 4.8σ shift when a_1 is large.

The results of this section indicate that the moving average control chart for the process mean is an economically viable alternative to the $\overline{X}$ chart for a variety of cases.

3.7 Startup Operation

All of the results discussed in this chapter have been obtained from an economic model which assumes that the moving average procedure is in steady state operation, and startups have not been allowed. In an earlier section, the assertion was made that this would not have a very great effect on the expected cost of using a moving average control chart. Now that numerical results have been obtained, it is possible to show that this is indeed the case. Consider Table 7 below. It shows the expected cost of using a moving average control chart in the steady state when the parameter values are those listed. The cost structure illustrated in Table 7 is typical. The lowest cost (\$.240) is obtained when the moving average index is four and the sample size is one, but when the moving average index is one, two or three, the cost is \$.242

Table 7. Moving Average Control Chart Cost, with Shifts of 2.4σ, α = .0001, λ' = 1000, a_1 = 1, a_2 = 1, a_3 = 100, a_4 = 10 and K = 25.

Sample Size	1	2	3	4	5	6
1	.242	.241	.241	.240	.240	.240
2	.282	.280	.280	.280	.280	.280

and \$.241, respectively. While the procedure is starting up (with a value of M less than four) the costs are not significantly greater than when the procedure is in steady state operation. In view of this it was felt that the omission of startup probability from the mathematical model was not significant.

CHAPTER IV

CONCLUSIONS AND RECOMMENDATIONS

4.1 Conclusions

In this investigation it has been assumed that the process to be controlled by the moving average chart has one in control state and one out of control state, and that shifts out of control follow the exponential distribution. The solution procedure described in Chapter III is used to develop an optimal set of parameters (M,N,K, and α) that will result in efficient quality control for such a process. The cost model developed on this basis can be applied to any number of production processes where these assumptions are valid. While the assumptions may tend to reduce the absolute accuracy of the model, it is felt that the simplicity and ease of application that result will outweigh this limitation.

When the moving average chart is applied to a number of experimental situations, the results indicate:

1) The cost of using the moving average chart is less than the cost of the $\overline{X}$ chart.

2) The size of the moving average index (M) is sensitive to the value of the critical region parameter (α) and to the size of the shift.

3) The optimal value of the sample size (N) is not influenced by changing any other system parameter.

4) The optimal value of the sampling interval (K) is sensitive to changes in a_1 (fixed cost of sampling), the size of the shift, and to the size of the mean deterioration rate (λ').

5) The optimal value of the critical region parameter is influenced by the cost of investigating and correcting (a_3).

6) The cost surface seems to be convex over a range of values of N and K near the optimal.

The results of this investigation tend to support Duncan's (4) claim of the importance of accuracy in the estimation for the model parameters and cost coefficients of the process under study. If this information is not known with certainty, the apparent accuracy of the model results are misleading.

4.2 Recommendations

The purpose of this section is to propose several topics related to this investigation which could be profitably pursued.

1) The results of this investigation could be extended to include other types of multi-period control charts, such as the geometric moving average chart.

2) The effect of increasing the number of out of control states could be investigated. While this study was concerned for a process with a single out of control state, some economic advantage or added realism might be gained by multiple states.

3) An automatic pattern search technique (such as the Hooke and Jeeves) could be employed in conjunction with the solution procedures described here to facilitate computation.

4) Startup probabilities could be explicitly considered in the mathematical model.

APPENDIX I

CONVERSION OF MODEL TO COMPUTER LANGUAGE

Computer Symbology

Model Symbol	Computer Symbol
μ_0	MUO
μ_1	MU1
σ	SIGMA
K1	K1
λ	LAMDA
$\underline{P}$	PBAR
P_0	po
P_1	p1
δ_0	DELTA
β	BETA
F	F
γ_0	GAMMA
α	AL
$Z\alpha/2$	ZAL
λ'	R
f_0	F0
f_1	F1
E(C)	TC

APPENDIX II

FORTRAN PROGRAM FOR NUMERICAL SOLUTION

```
      REAL MU0,MU1,    K1,LAMDA,LWRLIM
      DIMENSION M(11),N(11),K(26),PBAR(11,11,12),TC(11,11,26),P0(26),
     CP1(26),DTEST(100),       DIF(100),       DELTA(11,11,26),BETA(11,11
     C,26),F(26),GAMMA(11,11,99),AL(11),ZAL(11),      JSTEPS(26)
     C,A1COST(2),A2COST(2),A3COST(2),A4COST(2),RLEVEL(2)
C
C
C
C     EFFECTS  OF  R=10000  ON  SHIFTS  OF  2.4  SIGMA
C
C
C
C*****************************************************************
C   I1 IS THE INDEX ON M, THE NUMBER OF TERMS IN THE MOVING AVERAGE
C   I2 IS THE INDEX ON N, THE SAMPLE SIZE
C   I3 IS THE INDEX ON K, THE NUMBER OF UNITS BETWEEN SAMPLES
C*****************************************************************
      I1TOT=6
      I2TOT=2
      I3TOT=18
      AL(1)=.0001
      ZAL(1)=3.87
      AL(2)=.001
      ZAL(2)=3.29
      AL(3)=.002
      ZAL(3)=3.09
      AL(4)=.005
      ZAL(4)=2.81
      AL(5)=.01
      ZAL(5)=2.576
      AL(6)=.02
      ZAL(6)=2.33
      AL(7)=.05
      ZAL(7)=1.96
      AL(8)=.1
      ZAL(8)=1.65
      LAMDA=1.
      MU0=10.
      MU1=17.2
      SIGMA=3.
      R=10000.
```

```
      A1COST(1)=1.
      A1COST(2)=10.
      A2=1.
      A3COST(1)=10.
      A3COST(2)=100.
      A4=10.
      M(1)=1
      DO 5 I=1,I1TOT
      L=I+1
 5    M(L)=M(I)+1
      N(1)=1
      N(2)=2
      N(3)=3
      N(4)=5
      N(5)=10
      N(6)=20
      N(7)=30
      K(1)=90
      K(2)=100
      K(3)=110
      K(4)=130
      K(5)=140
      K(6)=150
      K(7)=170
      K(8)=180
      K(9)=260
      K(10)=270
      K(11)=290
      K(12)=300
      K(13)=310
      K(14)=330
      K(15)=340
      K(16)=350
      K(17)=370
      K(18)=380
      F0=.0027
      F1=.2742
C*****************************************************************
C  CALCULATE P0, THE PROBABILITY  THAT NO SHIFT OCCURS DURING THE
C  PRODUCTION OF K UNITS
C*****************************************************************
      DO 8 I=1,I3TOT
      P0(I)=1./EXP((LAMDA*K(I))/R)
C*****************************************************************
C*****************************************************************
      JSTEPS(I)=(P0(I)*100.)
      P1(I)=1.-P0(I)
 8    CONTINUE
C*****************************************************************
```

```
C   INCREMENT ON ALPH
C*****************************************************************************
      DO 300 IAL=1,8
      ALPH=AL(IAL)
      ZALPH2=ZAL(IAL)
C*****************************************************************************
C   CALCULATE PBAR(I), THE CONDITIONAL PROBABILITY OF MAKING A TYPE II ERROR
C*****************************************************************************
      DO 10 I1=1,I1TOT
      DO 10 I2=1,I2TOT
      I1PLUS=I1+1
      DO 10 I=1,I1PLUS
      X=I1
      K1=((((I1-I+1)*MU0+(I-1)*MU1)/X)-MU0)/(SIGMA/SQRT(X*N(I2)))
      LWRLIM=-ZALPH2-K1
      UPRLIM=ZALPH2-K1
      AREA1=RNORM(LWRLIM)
      AREA2=RNORM(UPRLIM)
      PBAR(I1,I2,I)=AREA2-AREA1
 10   CONTINUE
C*****************************************************************************
C   CALCULATE DELTA, THE STEADY STATE PROBABILITY OF BEING IN CONTROL ON
C   ANY TEST
C*****************************************************************************
      DO 117 I1=1,I1TOT
      DO 116 I2=1,I2TOT
      DO 115 I3=1,I3TOT
      DELTA(I1,I2,I3)=.01
      ISTEP=JSTEPS(I3)
      I1PLUS=I1+1
      DIFMIN=100.
      DO 114 J=1,100
 114  DTEST(J)=0.
      DO 113 L1=1,ISTEP
      DO 112 I =1,I1
      S1=(P0(I3)-DELTA(I1,I2,I3))/(P0(I3)-P0(I3)*DELTA(I1,I2,I3))
      S2=S1**(I -1)
      S3=PBAR(I1,I2,I1PLUS)*(P0(I3)**(I1-I))*P1(I3)*S2
      DTEST(L1)=DTEST(L1)+S3
 112  CONTINUE
      S5=S1**(I1-1)
      S6=(1.-(DELTA(I1,I2,I3)/P0(I3)))
      S7=P0(I3)*(1.-DELTA(I1,I2,I3))*PBAR(I1,I2,I1)*S6*S5-P0(I3)
      DTEST(L1)=-DELTA(I1,I2,I3)*(1.-DELTA(I1,I2,I3))*DTEST(L1)-S7
      DIF(L1)=ABS(DTEST(L1)-DELTA(I1,I2,I3))
      DELTA(I1,I2,I3)=DELTA(I1,I2,I3)+.01
C*****************************************************************************
C   FINDING THE BEST FIT FOR DELTA(I1,I2,I3)
C*****************************************************************************
      IF(DIF(L1)-DIFMIN)131,113,113
 131  DIFMIN=DIF(L1)
      L2=L1
 113  CONTINUE
      DELTA(I1,I2,I3)=DTEST(L2)
 115  CONTINUE
 116  CONTINUE
 117  CONTINUE
```

```
C*************************************************************************
C  CALCULATE BETA, THE PROBABILITY OF MAKING A TYPE II ERROR ON A TEST
C*************************************************************************
      DO 150 I1=1,I1TOT
      DO 150 I2=1,I2TOT
      DO 150 I3=1,I3TOT
      X12=(P0(I3)-DELTA(I1,I2,I3))/(P0(I3)-P0(I3)*DELTA(I1,I2,I3))
      BETA(I1,I2,I3)=X12
 150  CONTINUE
C
C************************************************************************
C  CALCULATE GAMMA, THE PROBABILITY OF BEING IN CONTROL AT ANY POINT
C************************************************************************
      DO 160 I3=1,I3TOT
      X1=(LAMDA*K(I3))/R
      F(I3)=(1.-((1.+X1)*EXP(-X1)))/(X1*(1.-EXP(-X1)))
 160  CONTINUE
      DO 170 I1=1,I1TOT
      DO 170 I2=1,I2TOT
      DO 170 I3=1,I3TOT
      X12=DELTA(I1,I2,I3)+F(I3)*(DELTA(I1,I2,I3)/P0(I3))*(1.-P0(I3)) ✓OK
      GAMMA(I1,I2,I3)=X12
 170  CONTINUE
C************************************************************************
C   CALCULATE  TC
C************************************************************************
      DO 800 INDXA1=1,2
      A1=A1COST(INDXA1)   not in
      DO 800 INDXA3=1,2     cards
      A3=A3COST(INDXA3)
      DO 190 I1=1,I1TOT
      DO 190 I2=1,I2TOT
      DO 190 I3=1,I3TOT
      X2=K(I3)
      X3=N(I2)
      Y6=(A1+A2*X3)/X2
      Y7=(1.-BETA(I1,I2,I3))*(1.-DELTA(I1,I2,I3))
      Y8=(A3/X2)*(ALPH*DELTA(I1,I2,I3)+Y7)
      Y9=(F1)*(1.-GAMMA(I1,I2,I3))
      Y10=A4*(F0*GAMMA(I1,I2,I3)+Y9)
      TC(I1,I2,I3)=Y6+Y8+Y10
 190  CONTINUE
C
C************************************************************************
C   PRINT  OUT  RESULTS
C************************************************************************
      DO 700 I3=1,I3TOT
      WRITE(6,1999)ALPH
 1999 FORMAT('0','ALPH=',    F7.6)
      WRITE(6,1994)R
 1994 FORMAT(' R=',    F10.1)
      WRITE(6,1995)A1
 1995 FORMAT(' A1=',F10.1)
      WRITE(6,1996)A2
 1996 FORMAT(' A2=',F10.1)
      WRITE(6,1997)A3
 1997 FORMAT(' A3=',F10.1)
```

```
      WRITE(6,1998)A4
 1998 FORMAT(' A4=',F10.1)
      WRITE(6,2000)K(I3)
 2000 FORMAT(' NUMBER OF UNITS BETWEEN SAMPLES IS'I4)
      WRITE(6,2001)(M(I1),I1=1,I1TOT)
 2001 FORMAT('0','MOVING AVG INDEX'2X,10I9)
      WRITE(6,2002)
 2002 FORMAT(' ',5X,'SAMPLE SIZE')
      DO 701 I2=1,I2TOT
      WRITE(6,2003)N(I2),(TC(I1,I2,I3),I1=1,I1TOT)
 2003 FORMAT(' ',I16,5X,10F9.3)
 701  CONTINUE
 700  CONTINUE
 800  CONTINUE
 300  CONTINUE
 710  CONTINUE
      STOP
      END
```

BIBLIOGRAPHY

1. Baker, Kenneth R., "Two Process Models in the Economic Design of an $\bar{X}$ Chart," AIIE Transactions, Vol. III, No. 4 (December, 1971), 257-263.

2. Cowden, D. J., Statistical Methods in Quality Control, Prentice-Hall Inc., Englewood Cliffs, New Jersey, 1957.

3. Duncan, A. J., Quality Control and Industrial Statistics, Third Edition, Richard D. Irvin, Inc., Homewood, Illinois, 1965.

4. Duncan, A. J., "The Economic Design of $\bar{X}$ Charts Used to Maintain Current Control of a Process," Journal of the American Statistical Association, Vol. 51, (1956) 228-242.

5. Duncan, Acheson J., "The Economic Design of $\bar{X}$ Charts When There is a Multiplicity of Assignable Causes," Journal of the American Statistical Association, Vol.

6. Gibra, I. N., "Economically Optimal Determination of the Parameters of an $\bar{X}$-Control Chart," Management Science, Vol. 17, No. 9 (May, 1971).

7. Goel, A. L. Jain, S. D. and Wu, S. M., "An Algorithm for the Determination of the Economic Design of $\bar{X}$-Charts Based on Duncan's Model," Journal of the American Statistical Association, Vol. 63, (1968) 304-320.

8. Grant, E. L., and Leavenworth, R. S., Statistical Quality Control, Third Edition, McGraw-Hill Book Company, New York, 1964.

9. Heikes, Russell G., Montgomery, Douglas C., and Yeung, Jimmy Y. H., "Alternative Process Models in the Economic Design of T^2 Control Charts," AIIE Transactions, Vol. VI, No. 1 (March, 1974).

10. Knappenberger, H. Allen, and Grandage, A. H. E., "Minimum Cost Quality Control Tests," AIIE Transactions, Vol. I, No. 1 (March, 1969), 24-32.

11. Mance, Joseph Frank, "Economic Design of Fraction Defective Control Charts to Maintain Current Control of a Process," M.S. Thesis, Georgia Institute of Technology, 1974.

12. Montgomery, Douglas C., and Klatt, Phillip, "Economic Design of T^2 Control Charts to Maintain Current Control of a Process," Management Science, Vol. 19, No. I (September, 1972), 76-89.

13. Roberts, S. W., "Control Charts Tests Based on Geometric Moving Averages," Techometrics, Vol. 1, No. 3 (August, 1959), 239-250.

14. Roberts, S. W., "A Comparison of Some Control Chart Procedures," Techometrics, Vol. 8, No. 3 (August, 1966), 411-430.

15. Shewhart, Walter A., Economic Control of Quality of Manufactured Product, D. Van Nostrand, Inc., Princeton, N. J., 1931.

16. Taylor, H. M., "Statistical Control of a Gaussian Process," Technometrics, Vol. 9, No. 1 (February, 1967), 29-41.

17. Wortham, A. W. and Heinrich, G. F., "Control Charts Using Exponential Smoothing Techniques," Transactions of the American Society of Quality Control, Washington, D. C., Spring 1972.